# Lecture no 29, 30 &31

## Definition of the *z*-Transform

- Given a finite length signal as

$x[n]$, the $z$-transform is defined

$$\boxed{X(z) = \sum_{k=0}^{N} x[k]z^{-k} = \sum_{k=0}^{N} x[k](z^{-1})^{k}} \quad (7.1)$$

  where the sequence support interval is [0, $N$], and $z$ is any complex number

- This transformation produces a new representation of $x[n]$ denoted $X(z)$
- Returning to the original sequence (*inverse z-transform*) $x[n]$ requires finding the coefficient associated with the $n$th power of $z^{-1}$

- Formally transforming from the time/sequence/$n$-domain to the z-domain is represented as

$$n\text{-Domain} \overset{z}{\leftrightarrow} z\text{-Domain}$$

$$x[n] = \sum_{k=0}^{N} x[k]\delta[n-k] \leftrightarrow X(z) = \sum_{k=0}^{N} x[k]z^{-k}$$

- A sequence and its $z$-transform are said to form a *$z$-transform pair* and are denoted

$$x[n] \overset{z}{\leftrightarrow} X(z) \tag{7.2}$$

  - In the sequence or $n$-domain the independent variable is $n$
  - In the $z$-domain the independent variable is $z$

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Example:

$x[n] =$
$\delta[n - n_0]$

- Using the definition

$X(z)$

- Thus,

$$= \sum_{k=0}^{N} x[k]z^{-k}$$

$$= \sum_{k=0}^{N} \delta[k - n_0]z^{-k}$$

$$= z^{-n_0}$$

$$\delta[n - n_0] \overset{z}{\leftrightarrow} z^{-n_0}$$

Example:

$x[n] =$

$2\delta[n] + 3\delta[n-1] + 5\delta[n-2] + 2\delta[n-3]$

- By inspection we find that

$X(z)$

$= 2 + 3z^{-1} + 5z^{-2} + 2z^{-3}$

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Example:

$X(z)$

$= 4 - 5z^{-2} + z^{-3} - 2z^{-4}$

- By inspection we find that

$$x[n] = 4\delta[n] - 5\delta[n-2] + \delta[n-3] - 2\delta[n-4]$$

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- What can we do with the $z$-transform that is useful?

# The *z*-Transform and Linear Systems

- The $z$-transform is particularly useful in the analysis and design of LTI systems

### The *z*-Transform of an FIR Filter

- We know that for any LTI system with input

$x[n]$

and

impulse response $h[n]$, the output is

$$y[n] = x[n] * h[n] \qquad (7.3)$$

- We are interested in the $z$-transform of FIR filter $h[n]$, where for an

$$h[n] = \sum_{k=0}^{M} b_k \delta[n-k] \qquad (7.4)$$

- To motivate this, consider the input

$$x[n] = z^n, \quad -\infty < n < \infty \tag{7.5}$$

- The output $y[n]$ is

$$y[n] = \sum_{k=0}^{M} b_k x[n-k] = \sum_{k=0}^{M} b_k z^{n-k} = \left( \sum_{k=0}^{M} b_k z^{-k} \right) z^n \tag{7.6}$$

$k = 0$

- The term in parenthesis is the $z$-transform of known as the *system function* of the FIR filter

$h[n]$, also

- Like

$H(e^{j\omega})$

was defined in Chapter 6, we define the system

$$\boxed{H(z) = \sum_{k=0}^{M} b_k z^{-k} = \sum_{k=0}^{M} h[k] z^{-k}}$$ function as

(7.7)

- The $z$-transform pair we have just established is

$$h[n] \overset{z}{\leftrightarrow} H(z)$$

$$\sum_{k=0}^{M} b_k \delta[n-k] \overset{z}{\leftrightarrow} \sum_{k=0}^{M} b_k z^{-k}$$

- Another result, similar to the frequency response result, is

$$y[n] = h[n]*z^n = H(z)z^n \qquad (7.8)$$

– Note if $z = e^{j\hat{\omega}}$, we in fact have the frequency response result of Chapter 6

- The system function is an $M$th degree polynomial in complex variable $z$
- As with any polynomial, it will have $M$ roots or *zeros*, that is there are $M$ values $z_0$ such that

$$H(z_0) = 0$$

– These $M$ zeros completely define the polynomial to within a gain constant (scale factor), i.e.,

$$H(z) = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M} = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) \cdots (1 - z_M z^{-1}) = \frac{(z - z_1)(z - z_2) \cdots (z - z_M)}{z^M}$$

where $z_k$, $k = 1, \ldots, M$ denote the zeros

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<u>Example</u>: Find the Zeros of

$$h[n] = \delta[n] + \frac{1}{6}\delta[n-1] - \frac{1}{6}\delta[n-2]$$

- The z-transform is

$$H(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

$$= \left(1 + \frac{1}{2}z^{-1}\right)\left(1 - \frac{1}{3}z^{-1}\right)$$

$$= \left(z + \frac{1}{2}\right)\left(z - \frac{1}{3}\right) / z^2$$

- The zeros of

$H(z)$

are -1/2 and +1/3

- The difference equation

$$y[n] = 6x[n] + x[n-1] - x[n-2]$$

has the same zeros, but a different scale factor;

proof:

## Properties of the $z$-Transform

- The z-transform has a few very useful properties, and its definition extends to infinite signals/impulse responses

### The Superposition (Linearity) Property

$$\boxed{ax_1[n] + bx_2[n] \overset{z}{\leftrightarrow} aX_1(z) + bX_2(z)}$$

proof

(7.9)

$X(z)$

$= \sum_{n=0}$

$(ax_1$

$[n] + bx_2$

$[n])z^{-1}$

$N1 = a \sum_{n=0}$

$x\,[n]z^{-1} + b$

$\sum_{n=0}^{N}$

$x\ [n]z^{-1}$

$2 = aX_1(z) + bX_2(z)$

**The Time-Delay Property**

$$x[n-1] \overset{z}{\leftrightarrow}$$

$z^{-1}X(z)$

(7.10)

and

$${}^{0}x[n-n\ ] \overset{z}{\leftrightarrow}$$

<u>proof</u>: Consider

$z^{-n_0}X(z)$

(7.11)

$_0$ $_1$ $^N$then

$X(z)$

$N$

$= \alpha \quad + \alpha\ z^{-1} + \cdots + \alpha \quad z^{-N}$

$x[n]\ =$

$$\sum_{k=0}$$

$\alpha_k\delta[n-k]$

Let

$= \alpha_0\delta[n] + \alpha_1\delta[n-1] + \cdots + \alpha_N\delta[n-N]$

$Y(z)$

so

$= z^{-1}X(z)$

$0 \quad 1 \quad N = \alpha \; z^{-1} + \alpha \; z^{-2} + \cdots + \alpha \quad z^{-N-1}$

$y[n] =$

$=$

$\alpha_0\delta[n-1] + \alpha_1\delta[n-2] + \cdots + \alpha_N\delta[n-N-1]$

$x[n-1]$

Similarly

$Y(z)$

$\Rightarrow y[n]$

$= z^{-n_0}X(z)$

$= x[n-n_0]$

**A General *z*-Transform Formula**

- We have seen that for a sequence $x[n]$ having support interval $0 \le n \le N$ the $z$-transform is

$$X(z) = \sum_{n=0}^{N} x[n]z^{-n} \qquad (7.12)$$

- This definition extends for doubly infinite sequences having support interval $-\infty \le n \le \infty$ to

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} \qquad (7.13)$$

  – There will be discussion of this case in Chapter 8 when we deal with infinite impulse response (IIR) filters

# The $z$-Transform as an Operator

Unit Delay

The $z$-transform can be considered as an operator.

### Unit-Delay Operator

$x[n]$ $x[n]$

$z^{-1}$

$y[n] =$

$x[n-1]$

- In the case of the unit delay, we observe that

$y[n]$

$= z^{-1}\{x[n]\}$

$= x[n-1]$

(7.14)

unit delay operator

which is motivated by the fact that

- Similarly, the filter

$Y(z)$

$= z^{-1}X(z)$

$$y[n] = x[n] - x[n-1]$$

can be viewed as the operator

since

$y[n] =$

$(1 - z^{-1})\{x[n]\}$

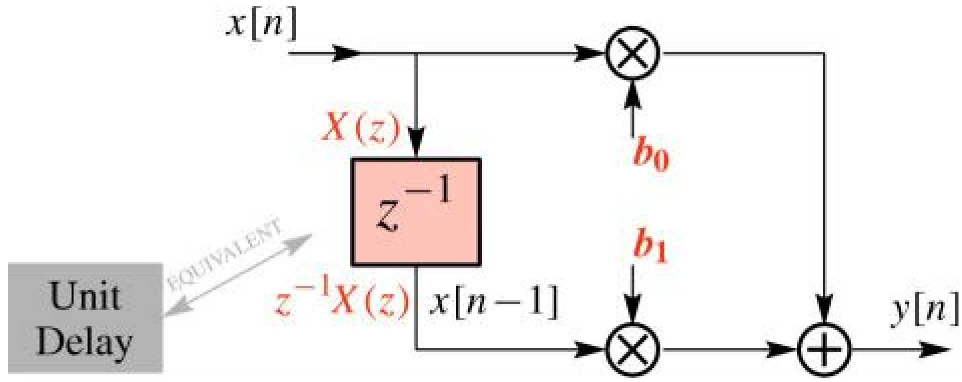
$= x[n] - x[n-1]$

$Y(z)$

$= X(z) - z^{-1}X(z)$

$= (1 - z^{-1})X(z)$

Example: Two-Tap FIR




- Using the operator convention, we can write by inspection that

$$Y(z) = b_0 X(z) + b_1 z^{-1} X(z)$$

$$y[n] = b_0 x[n] + b_1 x[n-1]$$

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# Convolution and the $z$-Transform

- The impulse response of the unity delay system is

$$h[n] = \delta[n-1]$$

and the system output written in terms of a convolution is

$$y[n] = x[n] * \delta[n-1]$$
$$= x[n-1]$$

- The system function (z-transform of $h[n]$) is

$$H(z) = z^{-1}$$

and by the previous unit delay analysis,

$$Y(z) = z^{-1}X(z)$$

- We observe that

$$\boxed{Y(z) = H(z)X(z)}$$

proof:

$M$

(7.15)

$$y[n] = x[n] * h[n] =$$

$$\sum_{k=0} h[k]x[n-k] \qquad (7.16)$$

We now take the $z$-transform of both sides of (7.16) using superposition and the general delay property

$$Y(z) = \sum_{k=0}^{M} h[k](z^{-k}X(z)) = \left(\sum_{k=0}^{M} h[k]z^{-k}\right)X(z) = H(z)X(z) \qquad (7.17)$$

- Note: For the case of

$x[n]$

a finite duration sequence,

$X(z)$ is

a polynomial, and

$z^{-1}$

$H(z)X(z)$

is a product of polynomials in

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<u>Example</u>: Convolving Finite Duration Sequences

- Suppose that

$x[n]$ $h[n]$

$= 2\delta[n] - 3\delta[n-2] + 4\delta[n-3]$

$= \delta[n] + 2\delta[n-1] + \delta[n-2]$

- We wish to find

$y[n]$

by first finding

$Y(z)$

- We begin by z-transforming each of the sequences

$X(z)$ $H(z)$

$= 2 - 3z^{-2} + 4z^{-3}$

$= 1 + 2z^{-1} + z^{-2}$

- We find

$Y(z)$

by direct multiplication

$Y(z)$

$= (2 - 3z^{-2} + 4z^{-3})(1 + 2z^{-1} + z^{-2})$

$= 2 + 4z^{-1} - z^{-2} - 2z^{-3} + 5z^{-4} + 4z^{-5}$

- We find
  $Y(z)$

$y[n]$

using the delay property on each of the terms of

$y[n]$

$= 2\delta[n] + 4\delta[n - 1] - \delta[n - 2]$

$- 2\delta[n - 3] + 5\delta[n - 4] + 4\delta[n - 5]$

Convolve directly?

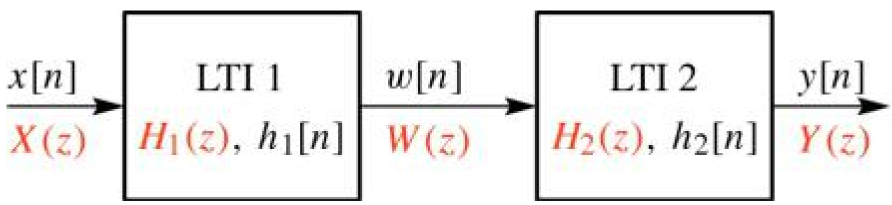
- This section has established the very important result that polynomial multiplication can be used to replace sequence convolution, when we work in the $z$-domain, i.e.,

**z-Transform Convolution Theorem**

$$y[n] = h[n]*x[n] \overset{z}{\leftrightarrow} H(z)X(z) = Y(z)$$

## Cascading Systems

- We have seen cascading of systems in the time-domain and the frequency domain, we now consider the $z$-domain




- We know from the convolution theorem that

- It also follows that

so by substitution

$W(z)$ $Y(z)$

$= H_1(z)X(z)$

$$= H_2(z)W(z)$$

$$Y(z)$$

$$= [H_2(z)H_1(z)]X(z)$$

$$= [H_1(z)H_2(z)]X(z)$$

$$(7.18)$$

- In summary, when we cascade two LTI systems, we arrive at the cascade impulse response as a cascade of impulse responses in the time-domain and a product of the z-transforms in the *z*-domain

$$\boxed{h[n] = h_1[n] * h_2[n] \overset{z}{\leftrightarrow} H_1(z)H_2(z) = H(z)}$$

**Factoring *z*-Polynomials**

- Multiplying *z*-transforms creates a cascade system, so factoring must create subsystems

Example:

- Since

$H(z)$ $H(z)$

$= 1 + 3z^{-1} - 2z^{-2} + z^{-3}$

______________________________________________________is a third-order polynomial, we should be able to
factor it into a first degree and second degree polynomial

- We can use the MATLAB function `roots()` to assist us

```
>> p = roots([1 3 -2 1])

p = -3.6274
     0.3137 + 0.4211i
     0.3137 - 0.4211i

>> conv([1 -p(2)],[1 -p(3)])

ans =    1.0000   -0.6274  0.2757 - 0.0000i
```

- With one real root, the logical factoring is to create two polynomials as follows

$H_1(z)\ H_2(z)$

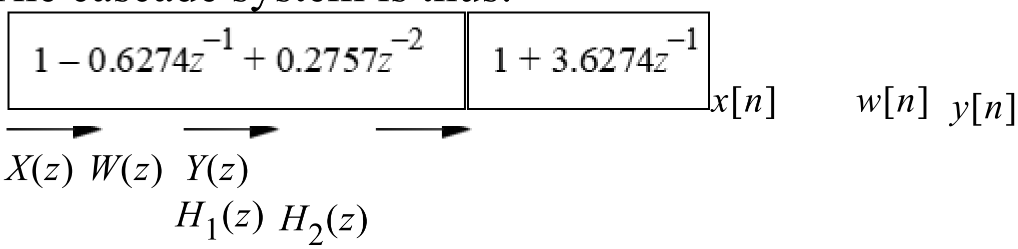
$= 1 + 3.6274z^{-1}$

$= \ (1 - (0.3137 + j0.4211)z^{-1})$

$(1 - (0.3137 - j0.4211)z^{-1})$

$= \ 1 - 0.6274z^{-1} + 0.2757z^{-2}$

- The cascade system is thus:




- As a check we can multiply the polynomials

```
>> conv([1 -p(1)],conv([1 -p(2)],[1 -p(3)]))

ans = 1.0000, 3.0000, -2.0000-0.0000i, 1.0000-0.0000i
```

- The difference equations for each subsystem are

$w[n]\ y[n]$

$= x[n] + 3.6274x[n-1]$

$= w[n] - 0.6274w[n-1] + 0.2757w[n-2]$

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**Deconvolution/Inverse Filtering**

- In a two subsystems cascade can the second system undo the action of the first subsystem?
- For the output to equal the input we need

- We thus desire

$H(z) = 1$

$$H_1(z)H_2(z)$$

$= 1$ or

$H_2(z) =$

$$\frac{1}{H_1(z)}$$

Example:

$H_1(z) = 1 - az^{-1}$, $a < 1$

- The inverse filter is

$$H_2(z) = \frac{1}{H_1(z)} = \frac{1}{1 - az^{-1}}$$

- This is no longer an FIR filter, it is an infinite impulse response (IIR) filter, which is the topic of Chapter 8
- We can approximate $H_2(z)$ as an FIR filter via long division

$$\begin{array}{r} 1 + az^{-1} + a^2 z^{-2} + \cdots \\ 1 - az^{-1} \overline{\big) \; 1 \qquad\qquad\qquad} \\ 1 - az^{-1} \end{array}$$

$$az^{-1}$$

$$az^{-1} - a^2 z^{-2}$$

$$a^2 z^{-2}$$

$$a^2 z^{-2} - a^3 z^{-3}$$

$$a^3 z^{-3}$$

- An

$M + 1$

term approximation is

$$H_2(z) = \sum_{k=0}^{M} a^k z^{-k}$$

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